

Dynamics of social contagions with limited contact capacity

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Individuals are always limited by some inelastic resources, such as time and energy, which restrict them to dedicate to social interaction and limit their contact capacities. Contact capacity plays an important role in dynamics of social contagions, which so far has eluded theoretical analysis. In this paper, we first propose a non-Markovian model to understand the effects of contact capacity on social contagions, in which each adopted individual can only contact and transmit the information to a finite number of neighbors. We then develop a heterogeneous edge-based compartmental theory for this model, and a remarkable agreement with simulations is obtained. Through theory and simulations, we find that enlarging the contact capacity makes the network more fragile to behavior spreading. Interestingly, we find that both the continuous and discontinuous dependence of the final adoption size on the information transmission probability can arise. There is a crossover phenomenon between the two types of dependence. More specifically, the crossover phenomenon can be induced by enlarging the contact capacity only when the degree exponent is above a critical degree exponent, while the final behavior adoption size always grows continuously for any contact capacity when degree exponent is below the critical degree exponent. © 2015 AIP Publishing LLC.

Studying models of social contagions provides insights into a variety of behavior spreading ranging from the adoption of an innovation and healthy activities to micro-finance. To date, most models for social contagions assume that individuals can contact all of neighbors during a short time. In reality, however, individuals exhibit limited contact capacity (i.e., individuals can only communicate or interact with a finite number of neighbors), due to the limitation of time, funds, energy, and other inelastic resources. In this paper, we introduce a non-Markovian behavior spreading model, in which adopted individuals can only transmit the information to limited number of neighbors and susceptible individuals become adopted only when their cumulative pieces of information rise above a threshold. We investigate the dynamics of our model on uncorrelated configuration networks and observe the limited contact capacity that suppresses the behavior spreading. In addition, we find a change of dependence of the final adoption size on the information transmission probability from being continuous to being discontinuous under some specific situations. We also provide a heterogeneous edge-based compartmental theory for solving the model that gives a good prediction for the qualities of the final behavior adoption. Our results offer some insight into understanding the effects of contact capacity on social contagions, and the developed theory has some references for establishing theoretical framework for other analogous dynamical processes.

I. INTRODUCTION

Humans are the basic constituents of the society, and every individual can interact with his/her family, friends, and peers. These interactions among individuals can induce some interesting collective behaviors, such as spontaneous formation of a common language or culture, emergence of consensus about a specific issue, and the adoptions of innovation, healthy, or microfinance behavior. Understanding the mechanisms or regularities behind these collective behaviors has led to a booming subfield of research in complex network science—social contagions, which has attracted much attention in recent years.^{1–3}

Statistical physics approaches were widely used to investigate social contagions. On the one hand, scientists used these methods to analyse large databases of social contagions and revealed that reinforcement effect widely exists.⁴ The reinforcement effect means that individual adopting a behavior is based on the memory of the cumulative behavioral information that he/she received from his/her neighbors. Centola^{5,6} established the artificially structured online communities to study health behavior spreading and found that the reinforcement effect significantly increases the adoption of a new health behavior. The reinforcement effect also exists in the adoptions of Facebook⁷ and Skype⁸ services. On the other hand, researchers proposed some novel models with reinforcement effect to describe the dynamics of social contagions. Among these models, linear threshold model^{9–11} is a famous one, and it is a deterministic model (i.e., a trivial case of Markovian process) once the network topology and initial seeds are fixed. In this model, an individual will adopt the behavior once the current fraction of his/her adopted neighbours is larger than a static threshold. The linear threshold model induces that the final behavior adoption size first

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grows continuously and then decreases discontinuously with the increasing of mean degree for vanishing small fraction of seeds. Another more realistic way to incorporate the reinforcement effect is whether an individual adopts the behavior should take his/her cumulative pieces of behavioral information into consideration.^{12–16} In this case, the dynamics is a non-Markovian process, which makes it more difficult to develop an accurate theory. Wang *et al.*¹⁵ proposed a non-Markovian behavior spreading model and found that the dependence of final behavior adoption size on information transmission probability can change from being discontinuous to being continuous under dynamical or structural parameters perturbation.

Recently, scholars found that individuals exhibit limited contact capacity (i.e., individuals can only communicate or interact with a finite number of neighbors during a short time) since the inelastic resources (e.g., time, funds, and energy) restrict them to dedicate to social interaction from empirical analysis.^{17–19} In the Facebook communication networks, Golder *et al.*²⁰ revealed that users only communicate with a small number of people although they have many declared friends. In the scientific cooperation networks, a scientist exchanges knowledge with only a fraction of his/her cooperators in a paper.^{21,22} In the sexual contact networks, individuals cannot have sexual intercourse with his/her all sexual partners in a very short time due to the limitation of morality and physiology.^{23,24} Researchers have studied the effects of contact capacity on some Markovian dynamics (i.e., epidemic spreading).^{25–27} They found that the epidemic outbreak threshold increases when the contact capacity is limited.²⁷ Meanwhile, each connection (edge) has distinct effective spreading probability (to be defined in Sec. III), which makes the theoretical prediction deviate from simulation results more easily, especially in the case of strong structural heterogeneity.

For the dynamics of social contagions, whether an individual adopts a behavior or not is determined by the cumulative pieces of behavioral information that he/she has received from neighbors.^{6,15} Once the contact capacity is limited, the behavioral information transmission will be limited, thus further affects the dynamics of social contagions. However, the systematic study to understand the effects of contact capacity on dynamics of social contagions is still lacking. In this paper, we try to address how the contact capacity affects the behavior spreading dynamics. We first propose a non-Markovian behavior spreading model with limited contact capacity, in which each adopted individual tries to transmit the behavioral information to limited number of his/her neighbors. In order to understand, quantitatively, the effects of contact capacity on the behavior spreading, we develop a heterogeneous edge-based compartmental theory. We find that the final behavior adoption size increases with the contact capacity. More interestingly, the crossover phenomenon is observed, which means that the dependence of the final adoption size on the information transmission probability can change from being continuous to being discontinuous. By enlarging the contact capacity, the crossover phenomenon can be induced only when the degree exponent is above a critical degree exponent. However, the final adoption size

always grows continuously for any contact capacity when degree exponent is below the critical degree exponent. The theoretical results from the suggested method can accurately predict the above results.

The paper is organized as follows. In Sec. II, we describe the behavior spreading model with limited contact capacity. We develop the heterogeneous edge-based compartmental theory in Sec. III. In Sec. IV, we verify the effectiveness of the theory through large number of simulations. Finally, we present conclusions and discussions in Sec. V.

II. BEHAVIOR SPREADING MODEL

We consider behavior spreading on uncorrelated configuration networks^{28,29} with N individuals (nodes) and degree distribution $P(k)$. We use a generalized stochastic SAR (susceptible-adopted-recovered) model¹⁵ to describe behavior spreading on networks. At each time step, each individual can be in one of the three different states: susceptible, adopted, or recovered. In the susceptible state, an individual does not adopt the behavior. In the adopted state, an individual adopts the behavior and tries to transmit the behavioral information (or simply information for short) to his/her selected neighbors. In the recovered state, an individual loses interest in the behavior and will not transmit the information further. Each individual holds a static adoption threshold κ , which reflects the wills (threshold) of an individual to adopt the behavior.

Initially, a fraction of ρ_0 individuals (nodes) are randomly selected to be in the adopted state (seeds), while other individuals are in the susceptible states. All susceptible individuals do not know any information about this behavior, in other words, the cumulative pieces of information is zero initially for all susceptible individuals. We denote the function $f(k')$ as the contact capacity of an adopted individual v , where k' is the degree of v . The larger value of $f(k')$, the more neighbors can receive the information from him/her. If $f(k') < k'$, the contact capacity of individual v is $f(k')$. If the contact capacity of v is larger than his/her degree [i.e., $f(k') \geq k'$], we let he/she transmit information to his/her all neighbors [i.e., $f(k') = k'$].

The synchronous updating method³⁰ is applied to renew the states of individuals. In this case, the time evolves discretely. At each time step, each adopted individual v with k' neighbors first randomly chooses a number of $f(k')$ neighbors due to the limited contact capacity and tries to transmit the information to each selected neighbor u . If individual u is in the susceptible state, u becomes adopted with probability λ ; otherwise, nothing happens. If v transmits the information to u successfully, the cumulative pieces of information m that u ever received will increase by 1, and the information cannot be transmitted between u and v in the following spreading process (i.e., redundant information transmission on the edge is forbidden). If m is larger than the adoption threshold κ , individual u becomes adopted in the next time step. From the mentioned procedures of susceptible individuals becoming adopted, we learn that the dynamics of social contagion is a non-Markovian stochastic process. Then, each adopted individual v loses interest in the

behavior and enters into recovered with probability γ . Individuals in the recovered state do not take part in the spreading process. The dynamics terminates once all adopted individuals become recovered.

III. HETEROGENEOUS EDGE-BASED COMPARTMENTAL THEORY

The non-Markovian behavior spreading model with limited contact capacity described in Sec. II makes theoretical prediction from the classical theory (e.g., heterogeneous mean-field theory) deviate from simulation results easily. On the one hand, in this proposed model, whether a susceptible individual adopts the behavior or not is dependent on the cumulative pieces of information he/she ever received. In this case, the memory effect of non-Markovian process is induced. On the other hand, the heterogeneity of effective spreading probability for edges increases with the heterogeneity of degree distribution and further enhances the difficulty in developing an accurate theory. The effective spreading probability of an edge includes two aspects: (1) an edge is randomly selected with probability $f(k')/k'$, where k' is the degree of adopted individual v ; (2) the information is transmitted through the selected edge with probability λ . Thus, the effective spreading probability of an edge for individual v is $\lambda f(k')/k'$.

To describe this process, we develop a heterogeneous edge-based compartmental theory, which is inspired by Refs. 31–33. In addition, the theory is based on the assumption that behavior spreads on uncorrelated and large sparse networks, we denote $S(t)$, $A(t)$, and $R(t)$ as the density of individuals in the susceptible, adopted, and recovered states at time t , respectively.

In the spirit of the cavity theory (i.e., message-passing approach),^{34,35} we let individual u in the cavity state (i.e., individual u cannot transmit information to his/her neighbors but can receive information from his/her neighbors). Denoting $\theta_{k'}(t)$ as the probability that an individual v with degree k' has not transmitted the information to individual u along a randomly selected edge up to time t . In heterogeneous networks, the adopted individuals are generally with different degrees. Thus, the values of $\theta_{k'}$ defined based on edges are heterogeneous, which are the naming refers of the heterogeneous edge-based compartmental theory. Considering all possible degrees of individual v , the average probability that individual u has not received the information from his/her neighbors by time t

$$\theta(t) = \sum_{k'=0} \frac{k'P(k')}{\langle k \rangle} \theta_{k'}(t), \quad (1)$$

where $k'P(k')/\langle k \rangle$ represents the probability that an edge from u connects to v with degree k' in uncorrelated network, and $\langle k \rangle$ is the mean degree. It is straightforward to get the probability that individual u with k neighbors has m cumulative pieces of information by time t

$$\phi(k, m, t) = (1 - \rho_0) \binom{k}{m} [\theta(t)]^{k-m} [1 - \theta(t)]^m. \quad (2)$$

The formula $1 - \rho_0$ represents that only individuals in the susceptible state initially can get the information. From

Sec. II, we know that only when u 's cumulative pieces of information are less than κ , he/she can be susceptible at time t . Thus, individual u is susceptible by time t with probability

$$s(k, t) = \sum_{m=0}^{\kappa-1} \phi(k, m, t). \quad (3)$$

Taking all possible values of k into consideration, we can get the fraction (density) of susceptible individuals at time t

$$S(t) = \sum_k P(k) s(k, t). \quad (4)$$

Similarly, we can get the fraction of individuals who have received m pieces of information at time t

$$\Phi(m, t) = \sum_{k=0} P(k) \phi(k, m, t). \quad (5)$$

According to the definition of $\theta_{k'}(t)$, one can further divide it as

$$\theta_{k'}(t) = \xi_{k'}^S(t) + \xi_{k'}^A(t) + \xi_{k'}^R(t). \quad (6)$$

The value of $\xi_{k'}^S(t)$, $\xi_{k'}^A(t)$, and $\xi_{k'}^R(t)$ represents that the probability of individual v with degree k' is susceptible, adopted, and recovered and has not transmitted information to his/her neighbors (e.g., individual u), respectively.

An initial susceptible neighbor individual v of u can only get the information from the other $k' - 1$ neighbors, since individual u is in the cavity state. Similar to Eq. (2), one can get the probability that v has m cumulative pieces of information by time t

$$\tau(k', m, t) = (1 - \rho_0) \binom{k' - 1}{m} [\theta(t)]^{k'-m-1} [1 - \theta(t)]^m. \quad (7)$$

We further get the probability of individual v in the susceptible

$$\xi_{k'}^S(t) = \sum_{m=0}^{\kappa-1} \tau(k', m, t), \quad (8)$$

which also reflects that the individual v remains susceptible only when his/her cumulative pieces of information are less than κ .

If the adopted neighbor individual v with degree k' transmits the information via an edge, this edge will not meet the definition of $\theta_{k'}(t)$. The conditions of individual v transmits information to u are (1) the edge connecting them is selected with probability $f(k')/k'$, which can approximatively reflect the limited contact capacity of v . Note that this approximation is more reasonable for larger k' . (2) The information is transmitted through this edge with probability λ . Thus, the evolution of $\theta_{k'}(t)$ is

$$\frac{d\theta_{k'}(t)}{dt} = -\frac{\lambda f(k')}{k'} \xi_{k'}^A(t). \quad (9)$$

If $f(k')$ is larger than k' , we restrict that v transmits the information to his/her all neighbors [i.e., $f(k') = k'$].

According to information spreading process described in Sec. II, the growth of $\xi_{k'}^R$ should simultaneously satisfy: (1) the adopted individual v does not transmit the information to u through the edge between them and (2) v moves into recovered state with probability γ . For the first condition, there are two possible cases: the edge between u and v is selected with probability $f(k')/k'$ and the information is not transmitted through it with probability $1 - \lambda$; the edge between u and v is not selected with probability $1 - f(k')/k'$. From the analyses above, the evolution of $\xi_{k'}^R$ is

$$\frac{d\xi_{k'}^R(t)}{dt} = \gamma \xi_{k'}^A(t) \left[1 - \frac{\lambda f(k')}{k'} \right]. \quad (10)$$

Now, combining Eqs. (9) and (10) and the initial situations [i.e., $\theta_{k'}(0) = 1$ and $\xi_{k'}^R(0) = 0$], we obtain the expression of $\xi_{k'}^R(t)$ in terms of $\theta_{k'}(t)$ as

$$\xi_{k'}^R(t) = \gamma [1 - \theta_{k'}(t)] \left[\frac{k'}{\lambda f(k')} - 1 \right]. \quad (11)$$

Utilizing Eqs. (6), (8), (9), and (11), we obtain that

$$\begin{aligned} \frac{d\theta_{k'}(t)}{dt} = & -\frac{\lambda f(k')}{k'} \left[\theta_{k'}(t) - \sum_{m=0}^{\kappa-1} \tau(k', m, t) \right] \\ & + \gamma [1 - \theta_{k'}(t)] \left[1 - \frac{\lambda f(k')}{k'} \right]. \end{aligned} \quad (12)$$

According to the model described in Sec. II, the densities of individuals in adopted and recovered individuals evolve as

$$\frac{dA(t)}{dt} = -\frac{dS(t)}{dt} - \gamma A(t) \quad (13)$$

and

$$\frac{dR(t)}{dt} = \gamma A(t), \quad (14)$$

respectively. Equations (4) and (13)–(14) give us a complete description of the social contagions with limited contact capacity. The evolution of each type of density versus time can be obtained.

The densities of susceptible, adopted, and recovered individuals do not change when $t \rightarrow \infty$. We denote $R(\infty)$ as the final behavior adoption size. To obtain the value of $R(\infty)$, one can first solve $\theta_{k'}(\infty)$ from Eq. (12), that is

$$\theta_{k'}(\infty) = \sum_{m=0}^{\kappa-1} \tau(k', m, \infty) + \gamma [1 - \theta_{k'}(\infty)] \left[\frac{k'}{\lambda f(k')} - 1 \right]. \quad (15)$$

Iterating Eq. (15) to obtain $\theta_{k'}(\infty)$. Then, inserting $\theta_{k'}(\infty)$ into Eqs. (1)–(4) to get the values of $S(\infty)$ and $R(\infty) = 1 - S(\infty)$.

Another important aspect we mainly focus on is the condition under which the global behavior adoption occurs. The global behavior adoption means that a finite fraction of individuals adopted the behavior, and the corresponding local

behavior adoption represents that only a vanishingly small fraction of individuals adopted the behavior. Similar to biological contagions, we define a critical transmission probability λ_c . When $\lambda \leq \lambda_c$, the behavior cannot be adopted by a finite fraction of individuals; when $\lambda > \lambda_c$, the global behavior adoption occurs. Now, we discuss λ_c for several different values of ρ_0 and κ .

For $\rho_0 \rightarrow 0$ (i.e., only a vanishingly small fraction of seeds) and $\kappa = 1$, $\theta_{k'}(\infty) = 1$ is the trivial solution of Eq. (15). If we change the values of other dynamical parameters, such as information transmission probability λ , a global behavior adoption may occur. The global behavior adoption occurs only when a nontrivial solution of Eq. (15) emerges [i.e., $\theta_{k'}(\infty) < 1$]. Note that the corresponding fraction of $\theta_{k'}(\infty)$ should be taken into consideration. Linearizing Eq. (15) at $\theta_{k'}(\infty) = 1$,²⁸ and summing all possible values of k' , one can get the critical information transmission probability

$$\lambda_c = \frac{\gamma \langle k \rangle G(k)}{\langle k^2 \rangle - (2 - \gamma) \langle k \rangle}, \quad (16)$$

where

$$G(k) = \sum_{k'} \frac{k'^2 P(k')}{\langle k \rangle f(k')}.$$

Note that λ_c is tightly correlated with the network topology [i.e., degree distribution $P(k)$] and dynamical parameters [i.e., contact capacity $f(k')$ and recover probability γ]. Hubs in heterogeneous networks adopt the behavior with large probability. Thus, λ_c decreases with the degree heterogeneity. The value of λ_c decreases with $f(k')$; in other words, increasing the contact capacity of individuals makes the network more fragile to the behavior spreading. We should emphasize that γ also affects λ_c/γ (i.e., the effective critical information transmission probability), which has been neglected in previous studies. The value of λ_c/γ increases with γ . If $f(k') \geq k'$ for every value of k' (i.e., adopted individual transmits the information to his/her all neighbors), Eq. (16) is the epidemic outbreak threshold.^{33,36} If every adopted individual only transmits information to his/her c neighbors, we can get the critical transmission probability λ_c of the model in Ref. 27.

For $\rho_0 \rightarrow 0$ and $\kappa > 1$, we find that $\theta_{k'}(\infty) = 1$ is the solution of Eq. (15). However, the left and right hands of Eq. (15) cannot be tangent to each other at $\theta_{k'} = 1$, which indicates that a vanishingly small seeds cannot trigger the global behavior adoption.¹⁵ With the increase of ρ_0 , different dependence of $R(\infty)$ on λ occurs for different κ . That is, the growth pattern of $R(\infty)$ versus λ can be continuous or discontinuous. Bifurcation theory is a widely used method to justify the type of phase transition.^{11,37} Through bifurcation analysis of Eq. (15), we find that $R(\infty)$ grows continuously for $\kappa = 1$, while a discontinuous growth may be induced for $\kappa > 1$.

The illustrations of the dependence of $R(\infty)$ on λ are presented in Fig. 1. For the case of $c = 1$, Eq. (15) has only one solution for any values of λ [see Fig. 1(a)]. In such a situation, $\theta(\infty)$ decreases continuously with λ [see the inset of Fig. 1(a)], which leads to a continuous growth of $R(\infty)$. For

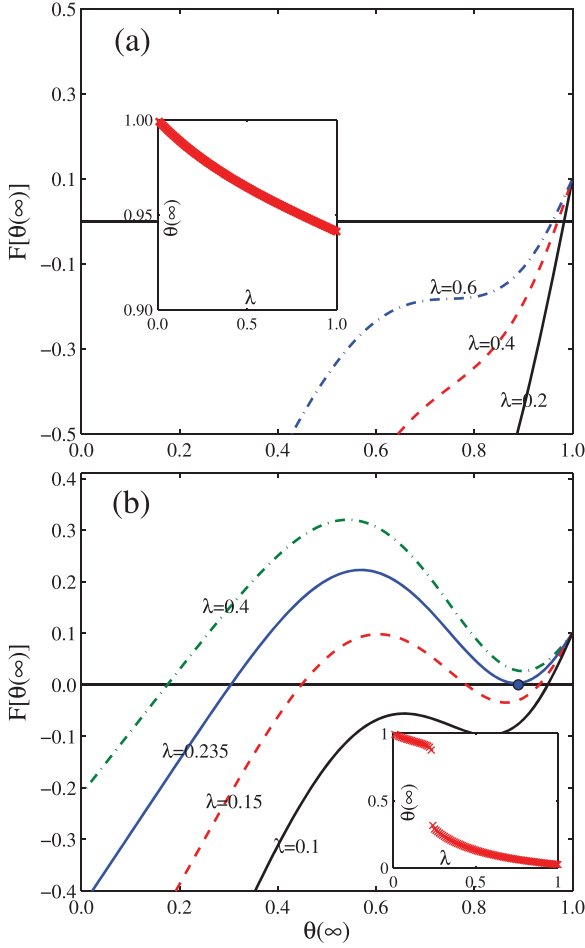


FIG. 1. Illustration of graphical solutions of Eq. (15). For random regular networks, (a) continuously increasing behavior of $R(\infty)$ with λ for $c=1$ and (b) discontinuous change in $R(\infty)$ for $c=8$. The black solid lines are the horizontal axis and the blue dot denotes the tangent point. The insets of (a) and (b) show the physically meaningful solutions of $\theta(\infty)$ versus λ for $c=1$ and $c=8$, respectively. The value of $F[\theta(\infty)]$ denotes $\theta(\infty) - \sum_{m=0}^{k-1} \tau(k, m, \infty) - \gamma[1 - \theta(\infty)] \left[\frac{k}{2f(k)} - 1 \right]$ and $f(k)=c$. Other parameters are set to be $\langle k \rangle = k = 10$, $\rho_0 = 0.1$, $\kappa = 3$, and $\gamma = 0.1$.

the case of $c=8$, the number of roots of Eq. (15) changes with λ [see Fig. 1(b)]. Numerical calculations indicate that the number of roots is either 1 or 3, which means that a saddle-node bifurcation occurs.³⁷ The bifurcation analysis of Eq. (15) reveals that the system undergoes a cusp catastrophe, i.e., the physically meaningful stable solution of $\theta(\infty)$ will suddenly jump to an alternate result by varying λ [see the inset of Fig. 1(b)]. In this situation, $R(\infty)$ increases discontinuously with λ . For small λ (e.g., $\lambda = 0.1$), the solution is the only fixed point. With the increase of λ , Eq. (15) has three fixed points. Note that only the maximum value of the stable fixed point (if there exist more than one stable fixed points, e.g., $\lambda = 0.15$) of Eq. (15) is physically meaningful in simulations, since $\theta(t)$ decreases with t . At the discontinuous information transmission probability $\lambda_c^I = 0.235$, the physically meaningful solution is the tangent point. For $\lambda > \lambda_c^I$, the solution of Eq. (15) changes abruptly to a small solution from a relatively large solution at $\lambda = \lambda_c^I$, which leads to a discontinuous change in $R(\infty)$. With the similar discussions, we can demonstrate the type of dependence and get the value of λ_c^I for other different situations.

IV. SIMULATION RESULTS

In this section, we verify the effectiveness of the heterogeneous edge-based compartmental theory developed in Sec. III by lots of simulations. For each network, we perform at least 2×10^3 times for a dynamic process and measure the final fraction of individuals in the recovered $[R(\infty)]$ and subcritical state $[\Phi(\kappa - 1, \infty)]$. These results are then averaged over 100 network realizations.

To build the network topology, we use the uncorrelated configuration model²⁹ according to the given degree distribution $P(k) \sim k^{-\nu}$ with maximal degree $k_{max} \sim \sqrt{N}$. There is no degree-degree correlations when N is very large. The heterogeneity of network increases with the decrease of ν . For the sake of investigating the effects of heterogeneous structural properties on the social contagions directly, the network sizes and mean degree are set to be $N = 10\,000$ and $\langle k \rangle = 10$, respectively. All individuals with different degrees have the same contact capacity $f(k)=c$ and recover probability $\gamma = 0.1$.

We first study the effects of the adoption threshold κ and contact capacity c on the final behavior adoption size $R(\infty)$ for strong heterogeneous networks in Fig. 2. We find that $R(\infty)$ decreases with the increase of κ , since individuals adopting the behavior need to expose more information. Once the contact capacity increases (i.e., c increases), individuals in adopted state will have more chances to transmit

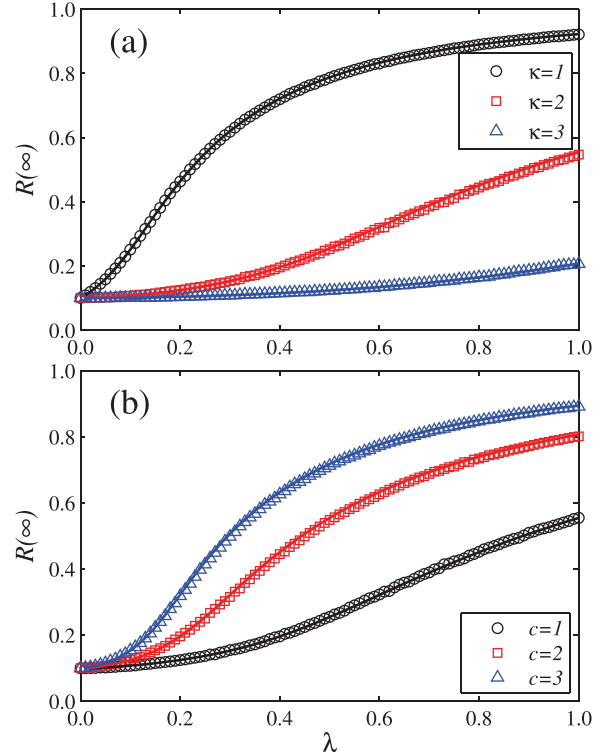


FIG. 2. On strong heterogeneous networks, the final adoption size $R(\infty)$ as a function of information transmission probability λ for (a) different adoption threshold κ and (b) different contact capacities c . In (a), black circles ($\kappa=1$), red squares ($\kappa=2$), and blue up triangles ($\kappa=3$) are the simulation results for $c=1$. In (b), black circles ($c=1$), red squares ($c=2$), and blue up triangles ($c=3$) are the simulation results for $\kappa=2$. In figure (a) and (b), the lines are the theoretical predictions from Eqs. (4) and (13)–(14). We set other parameters as $\nu = 2.1$, $\gamma = 0.1$, and $\rho_0 = 0.1$, respectively.

the information to susceptible individuals, thus, the values of $R(\infty)$ increases. Obviously, the theoretical predictions from heterogeneous edge-based compartmental theory agree well with the simulation results.

Another important issue we concern is the dependence of $R(\infty)$ on λ . As shown in Fig. 2, for strong heterogeneous networks, the dependence of $R(\infty)$ on λ is continuous for any values of κ and c , and we verify this claim by the bifurcation analysis of Eq. (15). We can also understand this phenomenon by discussing the fraction of individuals in the subcritical state from an intuitive perspective (see Fig. 3). An individual in the subcritical state means that he/she is in the susceptible state, and the m cumulative pieces of information is just one smaller than his/her adoption threshold κ . From Ref. 15, we know that the discontinuous dependence of $R(\infty)$ on λ will occur only when a large number of those subcritical individuals adopt the behavior simultaneously at some information transmission probability. Fig. 3 shows the final fraction of individuals in the subcritical state $\Phi(\kappa-1, \infty)$ versus λ . We find that $\Phi(\kappa-1, \infty)$ first increases and then decreases gradually with λ , since the existence of strong degree heterogeneity makes individuals in the subcritical state adopt the behavior consecutively. In these cases, a continuous growth of $R(\infty)$ versus λ occurs on strong heterogeneous networks.

We now study behavior spreading on weak heterogeneous networks, such as $\nu = 4.0$ in Fig. 4. Similar with the case of $\nu = 2.1$, increasing κ leads to the decrease of $R(\infty)$; and the value of $R(\infty)$ increases with c , that is the network will become more fragile to the behavior spreading once the contact capacity increases. Once again, our theory can predict the social dynamics very well. For the dependence of $R(\infty)$ on λ , a crossover phenomenon transition is observed. A crossover phenomenon means that the dependence of $R(\infty)$ on λ can change from being continuous to being discontinuous. More specifically, as shown in Fig. 4(b), the dependence of $R(\infty)$ on λ is continuous for small c (e.g., $c = 1$), while the dependence is discontinuous for larger c (e.g., $c = 8$). We justify this claim by the bifurcation analysis of Eq. (15) from the theoretical view, which is also verified through analyzing $\Phi(\kappa-1, \infty)$ from an

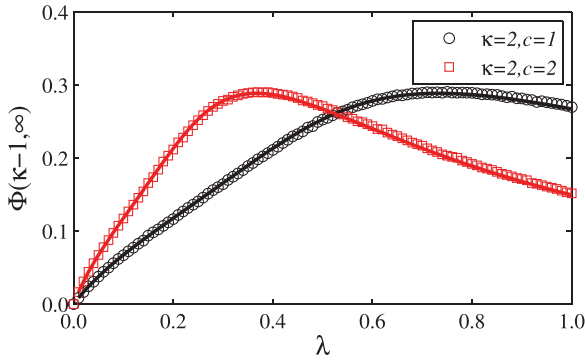


FIG. 3. The final fraction of individuals in the subcritical state $\Phi(\kappa-1, \infty)$ versus information transmission probability λ for $\kappa=2$, $c=1$ (black circles) and $\kappa=2$, $c=2$ (red squares). The lines are the theoretical predictions from Eqs. (5) and (13)–(14). Other parameters set to be $\nu = 2.1$, $\gamma = 0.1$, and $\rho_0 = 0.1$, respectively.

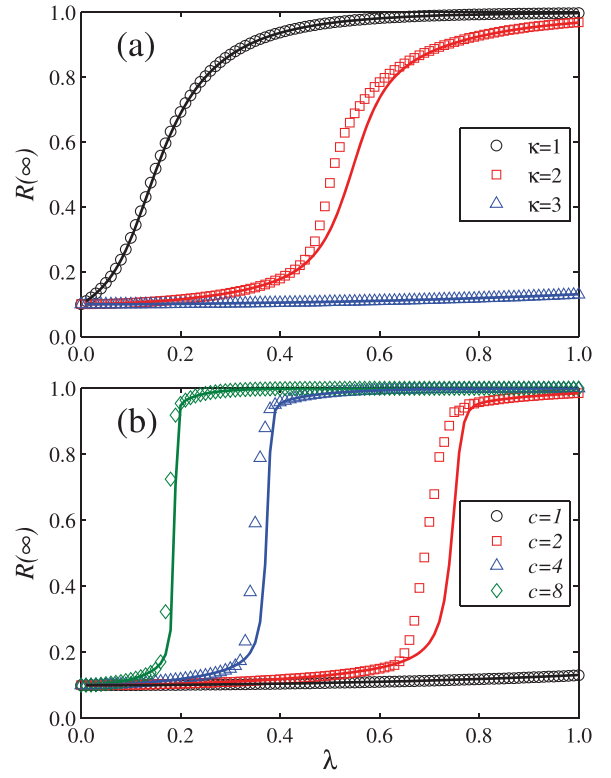


FIG. 4. On weak heterogeneous networks, the final adoption size $R(\infty)$ versus information transmission probability λ for (a) different adoption threshold κ and (b) different contact capacities c . In (a), black circles ($\kappa=1$), red squares ($\kappa=2$), and blue up triangles ($\kappa=3$) are the simulation results for $c=1$. In (b), black circles ($c=1$), red squares ($c=2$), blue up triangles ($c=4$), and green diamond ($c=8$) are the simulation results for $\kappa=3$. And the lines are the theoretical predictions, which are solved from Eqs. (4) and (13)–(14). We set other parameters as $\nu = 4.0$, $\gamma = 0.1$, and $\rho_0 = 0.1$.

intuitive perspective in Fig. 5. For weak heterogeneous networks, most individuals adopt the behavior with the same probability since they have similar degrees. When $c=1$, $\Phi(\kappa-1, \infty)$ increases continuously with λ , which leads to a continuous growth in the value of $R(\infty)$. When $c=8$, $\Phi(\kappa-1, \infty)$ first increases with λ and reaches a maximum at some values λ_c , and a slight increment of λ induces a finite fraction of $\Phi(\kappa, \infty)$ to adopt the behavior

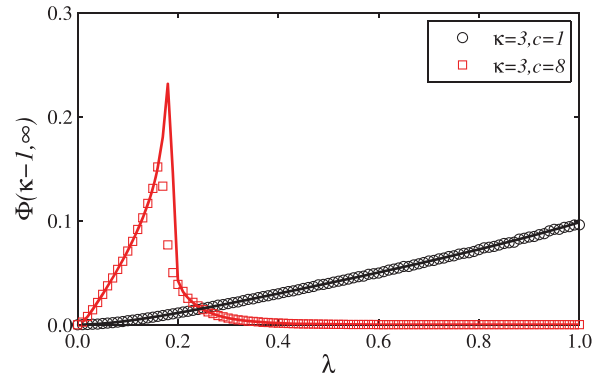


FIG. 5. The final fraction of individuals in the subcritical state $\Phi(\kappa-1, \infty)$ versus information transmission probability λ for $\kappa=3$, $c=1$ (black circles) and $\kappa=3$, $c=8$ (red squares). The lines are the theoretical predictions from Eqs. (5) and (13)–(14). Other parameters are $\nu = 4.0$, $\gamma = 0.1$, and $\rho_0 = 0.1$, respectively.

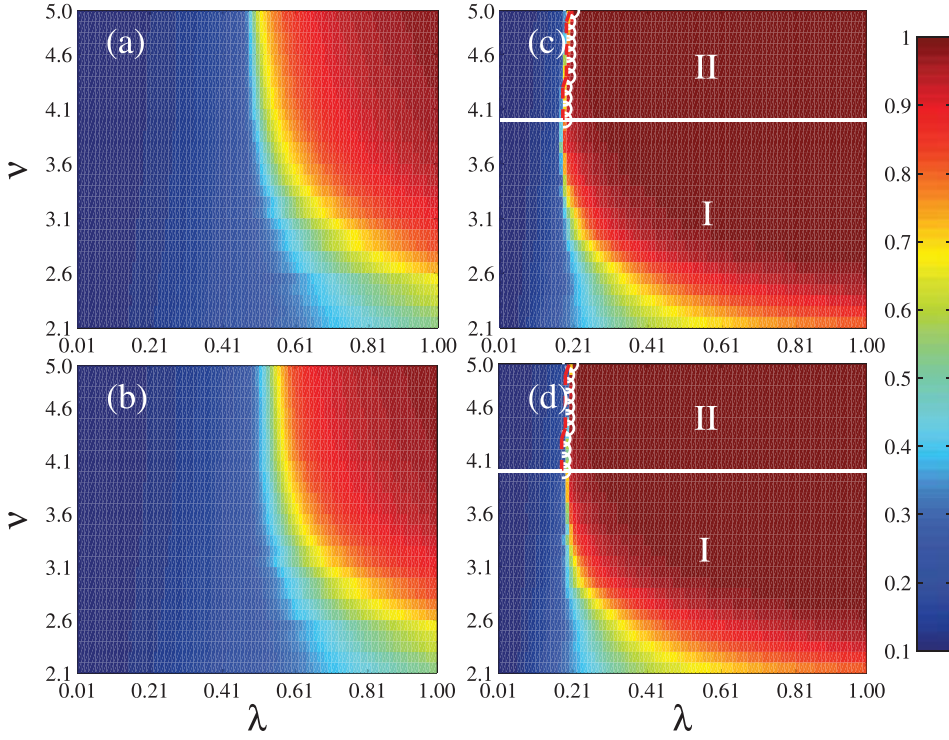


FIG. 6. The final behavior adoption size versus information transmission probability and degree exponent. (a) and (c) represent, respectively, the color-coded values of $R(\infty)$ from numerical simulations in the parameter plane λ - ν for $c=1$, $\kappa=2$ and $c=8$, $\kappa=3$. The theoretical predictions for $c=1$, $\kappa=2$ and $c=8$, $\kappa=3$ are shown in (b) and (d), respectively. And theoretical predictions are solved from Eqs. (4) and (13)–(14). In (b) and (d), $R(\infty)$ grows continuously with λ to a large value in region I. And in region II, $R(\infty)$ grows discontinuously and a finite fraction of individuals adopt the behavior above the discontinuous information transmission probability λ_c^I . The horizontal white line is the critical degree exponent ν_c , white circles and red dashed lines are simulated and theoretical results of λ_c^I , respectively. Other parameters are $\gamma = 0.1$ and $\rho_0 = 0.1$.

simultaneously, which leads to a discontinuous jump in the value of $R(\infty)$. We note that the crossover phenomenon in phase transitions of cascading failures can also arise in interdependent networks.^{38,39}

We further study the effects of ν and λ in Fig. 6 for different values of c . For the small contact capacity [i.e., $c=1$ in Figs. 6(a) and 6(b)], the dependence of $R(\infty)$ on λ is always continuous for any value of ν . In other words, this dependence is irrelevant to the network topology. For large contact capacity [i.e., $c=8$ in Figs. 6(c) and 6(d)], there is a crossover phenomenon in which the dependence of $R(\infty)$ on λ can change from being continuous to being discontinuous. More particularly, there is a critical degree exponent ν_c below which the dependence is continuous [see region I in Figs. 6(c) and 6(d)], while above ν_c the dependence is discontinuous [see region II in Figs. 6(c) and 6(d)]. The value of ν_c can be gotten by bifurcation analysis of Eq. (15). In region II, we also find that the discontinuous information transmission probability λ_c^I increases with ν , since the fraction of hubs decreases with ν . The theoretical predictions of λ_c^I can be gotten by bifurcation analysis of Eq. (15), and the simulation results of λ_c^I are predicted by NOI (number of iterations) method.¹⁵ Regardless of network heterogeneity, our theoretical predictions about the behaviors of $R(\infty)$ have a good agreement with numerical calculations. The average relative error⁴⁰ between the two predictions of $R(\infty)$ for all the values of λ and ν is less than 1.8%. For strong heterogeneous networks, some individuals with large degrees adopt the behavior more easily, thus makes the approximation of the edge's selecting probability $f(k')/k'$ in Eq. (9) be more reasonable. However, most of the adopted individuals have small degrees for less heterogeneous networks, thus makes the approximation be not very rigorous. As a result, the theoretical predictions agree well with the simulation results for

strong heterogeneous networks (e.g., $\nu = 2.1$ and $c=1$ in Fig. 7), whereas the deviation between the two predictions arises for less heterogeneous networks (e.g., $\nu = 4.0$ and $c=1$ in Fig. 7).

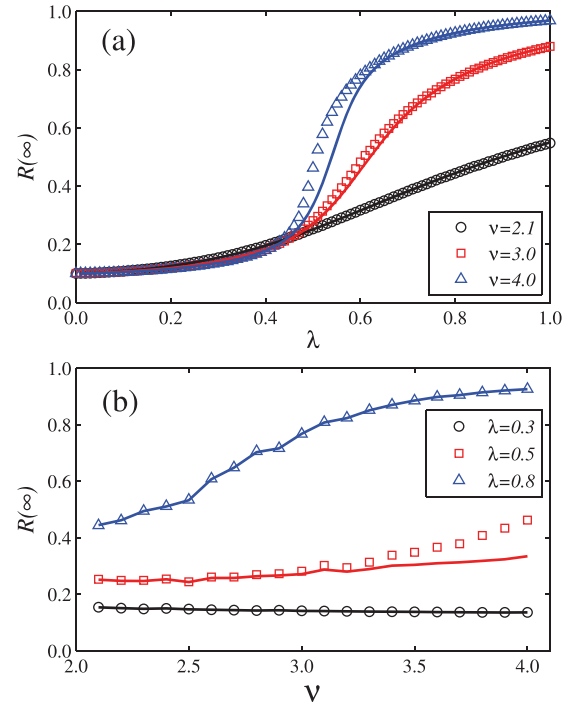


FIG. 7. Behavior spreading on scale-free networks. (a) The final behavior adoption size $R(\infty)$ versus information transmission probability λ for different degree exponents $\nu = 2.1$ (black circles), $\nu = 3.0$ (red squares), and $\nu = 4.0$ (blue up triangles). (b) The final behavior adoption size $R(\infty)$ versus $\lambda = 0.3$ (black circles), $\lambda = 0.5$ (red squares), and $\lambda = 0.8$ (blue up triangles), respectively. The lines are the theoretical predictions from Eqs. (4) and (13)–(14). Other parameters are $c=1$, $\gamma = 0.1$, $\kappa=2$, and $\rho_0 = 0.1$, respectively.

Finally, we study the effects of network topology on the final behavior adoption size $R(\infty)$ in Fig. 7 for $\kappa=2$ and $c=1$. We find that increasing ν can promote (suppress) behavior adoption at large (small) value of λ . This phenomenon can be qualitatively understood in the following ways:^{15,33} For the strong heterogeneous networks, many hubs and a large number of individuals with small degrees are coexisted. Since those hubs have more chances to expose the information, they adopt the behavior more easily even when λ is small. However, the situations for individuals with small degree are just opposite. That is, the large number of individuals with small degrees hinders the behavior adoption for large value of λ , thus, causes a smaller value of $R(\infty)$. Through bifurcation analysis of Eq. (15), the dependence of $R(\infty)$ on λ is always continuous for different ν .

V. DISCUSSION

To study social contagion dynamics in human populations is an extremely challenging problem with broad implications and interest. For social contagions on networks, some inelastic resources restrict individuals to dedicate to social interaction. This motivates the development of analytically tractable non-Markovian contagion models, in which individuals have limited contact capacity.

In this paper, we have proposed and analyzed such a model. Through theory and simulations, we illustrated that this model can exhibit interesting dynamics. The limited contact capacity suppresses the behavior spreading. In particular, we found a crossover phenomenon in which the dependence of $R(\infty)$ on λ can change from being continuous to being discontinuous. For uncorrelated configuration networks, a critical degree exponent ν_c is observed. When the degree exponent is above ν_c , the crossover phenomenon can be induced by enlarging c ; otherwise, $R(\infty)$ always grows continuously for any value of c .

In order to describe the non-Markovian characteristic and limited contact capacity by the heterogeneous edge-based compartmental theory, we made efforts from two aspects. On the one hand, in order to consider the non-Markovian characteristic, we first developed the meaning and content of $\theta_k(t)$ (i.e., the probability that an individual v with degree k has not transmitted the information to individual u in the cavity state along a randomly chosen edge by time t), then introduced the memory into Eqs. (2) and (7). On the other hand, we let the $\theta_k(t)$ and the effective spreading probability be heterogeneous to consider the limited contact capacity. The heterogeneous edge-based compartmental theory can predict the proposed model well. The average relative error between the theoretical predictions and numerical calculations is less than 1.8%.

In this work, we studied the behavior spreading with limited contact capacity theoretically and computationally, which is conducive to our understanding of the social contagions. Importantly, individual's contact capacity can alter the type of phase transition, which provides us a comprehensive understanding about the role of contact capacity and also changes the way that we think about

spreading dynamics. Here, we developed an accurate theoretical framework for non-Markovian social contagion model with limited contact capacity, which could be applied to other analogous dynamical processes (e.g., information diffusion⁴¹ and cascading⁴²). Furthermore, how to design an effective strategy to control the behavior spreading with limited contact capacity is an interesting research topic.

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